THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010H University Mathematics 2014-2015
Suggested Solution to Test 1

1. (a) $\lim_{n\to\infty} \frac{4n^2+3}{-n^2-5n+2} = \lim_{n\to\infty} \frac{4-\frac{3}{n^2}}{-1-\frac{5}{n}+\frac{2}{n^2}} = -4$

   (b) $\lim_{n\to\infty} \left(1+\frac{1}{n+1}\right)^{2n} = \lim_{n\to\infty} \left[\left(1+\frac{1}{n+1}\right)^{n+1}\right]^2 \cdot (1+\frac{1}{n+1})^{-2} = e^2\cdot 1 = e^2$

2. Note that $\frac{1}{\sqrt[4]{n^4+n}} \le \frac{1}{\sqrt[4]{n^4+i}} \le \frac{1}{\sqrt[4]{n^4}} = \frac{1}{n}$ for all $1\le i\le n$, so we have

$$\frac{1}{\sqrt[4]{n^4+n}}\cdot n \le \frac{1}{\sqrt[4]{n^4+1}} + \frac{1}{\sqrt[4]{n^4+2}} + \cdots + \frac{1}{\sqrt[4]{n^4+n}} \le \frac{1}{n}\cdot n = 1$$

Note that $\lim_{n\to\infty} \frac{n}{\sqrt[4]{n^4+n}} = 1$. By sandwich theorem,

$$\lim_{n\to\infty} \frac{1}{\sqrt[4]{n^4+1}} + \frac{1}{\sqrt[4]{n^4+2}} + \cdots + \frac{1}{\sqrt[4]{n^4+n}} = 1.$$

3. (a) $\lim_{x\to 0} \frac{\sin 2x \tan 3x}{x^2} = \lim_{x\to 0} \frac{\sin 2x}{2x}\cdot\frac{\sin 3x}{3x}\cdot\frac{6}{\cos 3x} = 1\cdot 1\cdot 6 = 6$

   (b) $\lim_{x\to+\infty} \frac{e^{x+1}+e^{-x}}{e^{x-1}-e^{-x}} = \lim_{x\to+\infty} \frac{e+e^{-2x}}{e^{-1}-e^{-2x}} = e^2$

   (c) $\lim_{x\to-\infty} \frac{x}{\sqrt{9x^2+1}} = \lim_{x\to-\infty} \frac{1}{\frac{1}{x}\sqrt{9x^2+1}} = \lim_{x\to-\infty} \frac{1}{-\sqrt{9+\frac{1}{x^2}}} = -\frac{1}{3}$

4. (a) $\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{0-0}{h} = 0$ and $\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{-h^3-0}{h} = 0$.
   Therefore, $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ exists and equals to 0, that means f is differentiable at $x=0$ and $f'(0)=0$.

   (b) If $x>0$, $f'(x)=0$. If $x<0$, $f'(x)=-3x^2$. Combine them with the result in (a), we have

$$f'(x) = \begin{cases} 0 & \text{if } x\ge 0 \\ -3x^2 & \text{if } x<0 \end{cases}$$

   We have $\lim_{h\to 0^+} \frac{f'(0+h)-f'(0)}{h} = \lim_{h\to 0^+} \frac{0-0}{h} = 0$, and $\lim_{h\to 0^-} \frac{f'(0+h)-f'(0)}{h} = \lim_{h\to 0^-} \frac{-3h^2-0}{h} = 0$. Therefore, $\lim_{h\to 0} \frac{f'(0+h)-f'(0)}{h} = 0$, i.e. f' is differentiable at $x=0$.

5. Let $f(x)=e^x$, so f is differentiable everywhere.
   If $x>y>0$, by Mean Value Theorem, there exists $c\in(y,x)$ such that

$$\begin{aligned} \frac{f(x)-f(y)}{x-y} &= f'(c) \\ e^x - e^y &= e^c(x-y) \end{aligned}$$

   Note that $x>c>y$, so $e^x>e^c>e^y$, and we have

$$e^y(x-y) < e^x - e^y < e^x(x-y).$$

6. (a) i. Put $x = y = 0$, we have $f(0) = [f(0)]^2$, so $f(0) = 0$ or 1. If we put 0 to the inequality in the second condition, we have $1 \leq f(0)$, so $f(0) = 1$.

ii. If $x < 0$, by the inequality in the second condition, we have

$$f(x) \geq 1 - x > 1.$$

iii. If $x > 0$, then

$$1 = f(0) = f(x - x) = f(x)f(-x).$$

Therefore, $f(x) = \dfrac{1}{f(-x)} > 0$.

If $a > b$, then

$$\begin{aligned} f(a) - f(b) &= f(a) - f(a + (b - a)) \\ &= f(a) - f(a)f(b - a) \\ &= f(a)(1 - f(b - a)) \\ &< 0 \end{aligned}$$

Note: $b - a < 0$, so $f(b - a) > 1$.

(b) We put h to the inequality in the second condition, we have

$$1 - h \leq f(h) \leq 1 - hf(h).$$

Also, $f(h) \leq 1 - hf(h)$ implies that $f(h) \leq \dfrac{1}{1 + h}$ if $h > -1$.
Therefore, when $h > -1$,

$$1 - h \leq f(h) \leq \frac{1}{1 + h}.$$

Note that $\lim\limits_{h \to 0} 1 - h = \lim\limits_{h \to 0} \dfrac{1}{1 + h} = 1$, so by sandwich theorem, we have

$$\lim_{h \to 0} f(h) = 1 = f(0),$$

which implies f is continuous at $x = 0$.

(c) From the inequality in the second condition, we have

$$\begin{aligned} 1 - h \leq &\quad f(h) &\leq 1 - hf(h) \\ -h \leq &\quad f(h) - 1 &\leq -hf(h) \end{aligned}$$

If $h > 0$, we have $-1 \leq \dfrac{f(h) - 1}{h} \leq -f(h)$ so by sandwich theorem, we have

$$\lim_{h \to 0^+} \frac{f(h) - 1}{h} = -1.$$

Similarly, if $h < 0$, we have $-1 \geq \dfrac{f(h) - 1}{h} \geq -f(h)$ so by sandwich theorem, we have

$$\lim_{h \to 0^-} \frac{f(h) - 1}{h} = -1.$$

Therefore, $\lim_{h \to 0} \frac{f(h) - 1}{h} = -1$.

Now, we have

$$\begin{aligned} \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} &= \lim_{h \to 0} f(0) \cdot \frac{f(h) - 1}{h} \\ &= -1 \end{aligned}$$

which implies that f is differentiable at $x = 0$ and $f'(0) = -1$.